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ON THE STRUCTURE OF ZEROFREE SEQUENCES

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1. Introduction

Let G be a finite abelian group, S a sequence in G and $\Sigma(S)$ the set of sums of non-empty subsequences of S. A huge variety of representation problems for special classes of sequences or subsets of G have been investigated. Thereby special emphasis has been laid on zerofree sequences i.e., sequences with $0 \notin \Sigma(S)$. Of course all kinds of representation problems for zerofree sequences (resp. zerofree subsets) are highly related to the structure of such sequences. In this paper we deal with the following two problems.

Firstly, what can be said about the maximal multiplicity of elements in long zerofree sequences? This problem was first tackled for cyclic groups by Erdős et al. in the seventies [1]. In particular, this is related to a question of Erdős and Heilbronn (cf. section 3) who asked for the maximal size of zerofree subsets in G (cf. [3] and the paper of Hamidoune and Zémor [6] for recent progress). We prove a result for general finite abelian groups which enbetters the old result if the sequence is not too long (Theorem 3.4 and Corollary 3.5).

Secondly, what can be said about the order of elements in long zerofree sequences? A current motivation for this question stems from the theory of non-unique factorization (the interested reader is referred to [5], Theorem 5.5). Furthermore, a recent conjecture states that every minimal zero sequence S in a finite abelian group G with length $|S| = D(G)$, where $D(G)$ is Davenport's constant, contains an element whose order equals the exponent of G (cf. section 6 in [4]). Using our answer to the first question we show that every zerofree sequence S in a cyclic group of order n with $|S| \geq \frac{n+3}{2}$ contains an element of order n (see Theorem 4.1).

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2. Preliminaries

Notation and terminology will be consistent with the one used in [4]. For convenience we give a brief review. Throughout, finite abelian groups will be written additively and for $n\in\mathbb{N}_+$ let C_n denote the cyclic group with n elements.

Let G be a finite abelian group. Then $\mathcal{F}(G)$ denotes the free abelian monoid with basis G, the elements of which are called *sequences* (in G). Let $S=\prod_{\nu=1}^{l} g_\nu$ be a sequence. Then it has a unique representation in the form

$$S = \prod_{g\in G} g^{v_g(S)} \ \in \ \mathcal{F}(G)$$

where $v_g(S)\in\mathbb{N}$ is called the *multiplicity* of $g\in S$. We say that S contains some $g\in G$, if $v_g(S)\geq 1$. Furthermore, $|S| = l = \sum_{g\in G} v_g(S)\in\mathbb{N}$ is called the *length* of S. The *empty sequence* is just the identity in $\mathcal{F}(G)$ and has length zero. A sequence $T\in\mathcal{F}(G)$ is called a *subsequence* of S, if $v_g(T)\leq v_g(S)$ for every $g\in G$. The monoid homomorphism $\iota:\mathcal{F}(G)\to G$ maps S to the sum of its elements i.e., $\iota(S)=\sum_{\nu=1}^{l} g_\nu=\sum_{g\in G} v_g(S)g\in G$. As usual,

$$\Sigma(S) = \{\iota(T) \mid T \ \text{ a non-empty subsequence of } S\}$$

denotes the set of sums of non-empty subsequences of S. We say that the sequence S is

squarefree, if $v_g(S)\leq 1$ for every $g\in G$,

zerofree, if $0\notin\Sigma(S)$,

a zero sequence, if $\iota(S)=0$.

a minimal zero sequence, if it is a zero sequence and each proper subsequence is zerofree.

Davenport's constant $D(G)$ is defined as

$$D(G) = \max\{|S| \mid S \text{ is a minimal zero sequence in } G\}\in\mathbb{N}_+.$$

A group homomorphism $\varphi: G\to H$ extends to a monoid homomorphism $\varphi:\mathcal{F}(G)\to\mathcal{F}(H)$ with $\varphi(S)=\prod_{\nu=1}^{l}\varphi(g_\nu)$.

3. On the maximal multiplicity of elements

Let G be a finite abelian group and $S=\prod_{\nu=1}^{l} g_\nu^{k_\nu}$ a zerofree sequence with pairwise distinct elements g_ν and with $k_1\geq\ldots\geq k_l\geq 1$. We try to find lower bounds for k_1, the maximal multiplicity of an element occuring in S. The groups we have in mind are groups where the exponent is large in comparision to the group order, hence in particular cyclic groups. For general groups nothing more than $k_1\geq 1$ can be achieved (cf. [4]).

To get a first rough bound, let us recall that there is some $c \le 3$ such that every squarefree zerofree sequence $T \in \mathcal{F}(G)$ has length $|T| \le c\sqrt{|G|}$ (This is due to Olson [8]; cf. the introduction in [6]). Therefore, we have $l \le c\sqrt{|G|}$ and $|S| \le k_1 l$ implies that

$$k_1 \ge \frac{|S|}{c\sqrt{|G|}} .$$

Hence, if $G = C_n$ and $|S| \ge \frac{n+3}{2}$ than $k_1 \ge \frac{n+3}{4\sqrt{n}}$. The following approach will yield a substantially better result (cf. Corollary 3.5).

Lemma 3.1. *Let G be a finite abelian group.*

1. *Let $A, B \subseteq G$ with $0 \in A$, $0 \in B$ be subsets such that $a+b=0$, $a \in A$, $b \in B$ implies that $a=b=0$. Then $|A+B| \ge |A|+|B|-1$.*
2. *Let $S \in \mathcal{F}(G)$ be a zerofree sequence, $r \in \mathbb{N}_+$ and $S_1, \dots, S_r$ non-empty sequences such that $S = S_1 \dots S_r$. Then $|\Sigma(S)| \ge \sum_{i=1}^{r} |\Sigma(S_i)|$.*

Proof. 1. see [7].

2. Obviously, it suffices to prove the assertion for $r = 2$. Then the general case follows by induction. Hence suppose $S = S_1 S_2$. Set $A = \Sigma(S_1) \cup \{0\}$ and $B = \Sigma(S_2) \cup \{0\}$. Then $A + B \setminus \{0\} \subseteq \Sigma(S_1 S_2)$ and 1) implies that

$$|\Sigma(S_1 S_2)| \ge |A+B| - 1 \ge |A| + |B| - 2 = |\Sigma(S_1) + |\Sigma(S_2)| \qquad \blacksquare$$

Let G be a finite abelian group, $\emptyset \ne G_0 \subseteq G$ a subset and $k \in \mathbb{N}_+$. Define

$$f(G_0, k) = \min\{|\Sigma(S)| \big| S \in \mathcal{F}(G_0) \text{ is a squarefree, zerofree sequence with } |S| = k\}$$

and set $f(G_0, k) = \infty$, if there are no sequences in G_0 of the above form. By definition, we have $f(G_0, k) \ge k$.

Lemma 3.2. *Let G be a finite abelian group and $k \in \mathbb{N}_+$.*

1. *$f(G,1)=1$ and $f(G,2)=3$,*
2. *$f(G,3) \ge 5$,*
3. *If G is cyclic and $G_0 \subseteq G$ a subset not containing an element of order 2, then $f(G_0, 3) \ge 6$,*
4. *$f(G,k) \ge 2k$ for $k \ge 4$,*
5. *$f(G,k) > \frac{1}{9}k^2$.*

Proof. 1. Obvious.

2. and 4. see Theorem 4 and 5 in [2].

3. see Lemma 2 in [1].

5. see Theorem 3.2 in [8]. $\blacksquare$

The following result was proved by Bovey, Erdős and Niven in [1].

Lemma 3.3. *Let $G = C_n$ with $n \ge 4$ and $S \in \mathcal{F}(G)$ a zerofree sequence with $|S| \ge \frac{n+1}{2}$. Then there exists some $g \in G$ with $v_g(S) \ge 2|S| - n + 1$.*

Theorem 3.4. *Let G be a finite abelian group, $G_0 \subseteq G$ a subset, $k \in \mathbb{N}_+$ and $S \in \mathcal{F}(G_0)$ a zerofree sequence. If $|S| \ge \big(\frac{|G|-k}{f(G_0,k)} + 1\big)k$, then there exists some $g \in G$ with*

$$v_g(S) \ge \frac{|S|}{k-1} - \frac{|G| - k - 1}{(k-1)f(G_0,k)}.$$

Proof. Let $t \in \mathbb{N}_+$ be maximal such that there are t squarefree sequences $A_1, \ldots, A_t$ with $k = |A_1| = \ldots = |A_t|$ and

$$S = A_1 \ldots A_t \cdot T$$

where $T = \prod_{i=1}^d g_i^{u_i}$ with $u_1 \ge \ldots \ge u_d \ge 1$ and $g_1, \ldots, g_d$ pairwise distinct. We fix a choice of $A_1, \ldots, A_t$, T such that d is maximal.

If $t \ge \frac{|G|}{f(G_0,k)}$, then by Lemma 3.1

$$|\Sigma(S)| \ge \sum_{i=1}^t |\Sigma(A_i)| \ge t f(G_0,k) \ge |G|,$$

a contradiction to the zerofreeness of S.

If $t = \frac{|G|-m}{f(G_0,k)}$ with $1 \le m \le k$, then

$$|T| = |S| - kt \ge \left(\frac{|G|-k}{f(G_0,k)} + 1\right)k - \frac{|G|-m}{f(G_0,k)}k = \left(1 - \frac{k-m}{f(G_0,k)}\right)k$$

whence

$$\begin{aligned} |\Sigma(S)| &\ge \sum_{i=1}^t |\Sigma(A_i)| + |\Sigma(T)| \\ &\ge t f(G_0,k) + |T| \\ &\ge |G| - m + \left(1 - \frac{k-m}{f(G_0,k)}\right)k \\ &\ge |G| + (k-m)\left(1 - \frac{k}{f(G_0,k)}\right) \ge |G|, \end{aligned}$$

a contradiction. Therefore we infer that

$$t f(G_0,k) \le |G| - k - 1.$$

Assertion. Let $i \in \{1, \ldots, d\}$ with $u_i \ge 2$. Then $v_{g_i}(A_j) \ge 1$ for every $1 \le j \le t$.

Proof. Assume to the contrary, that there is some $j \in \{1,\dots,t\}$ with $v_{g_i}(A_j)=0$; say $j=1$. The maximality of t implies that $d<k=|A_1|$. Hence there is some $g\in G$ with $v_g(A_1)\geq 1$ but $g\notin\{g_1,\dots,g_d\}$. Defining

$$A_1{}' = g^{-1}g_iA_1$$

we infer that

$$S = A_1{}'A_2\dots A_t g_1^{u_1}\dots g_{i-1}^{u_{i-1}}g_i^{u_i-1}g_{i+1}^{u_{i+1}}\dots g_d^{u_d}\cdot g,$$

a contradiction to the maximality of d.

Suppose that

$$u_1\geq\dots\geq u_h\geq 2 \quad\text{and}\quad u_{h+1}=\dots=u_d=1.$$

Since $d<k$ and

$$|T| = |S|-kt \geq \Big(\frac{|G|-k}{f(G_0,k)}+1\Big)k - k\Big(\frac{|G|-k-1}{f(G_0,k)}\Big) > k,$$

it follows that $h\geq 1$. Setting

$$\tau = \sum_{i=1}^{h} v_{g_i}(S)$$

the assertion implies that

$$\tau \geq ht + \sum_{i=1}^{h} u_i.$$

Therefore there is some $g\in\{g_1,\dots,g_h\}$ with

$$\begin{aligned}
v_g(S) \geq \frac{\tau}{h} &\geq \frac{1}{h}\left(ht+\sum_{i=1}^{h}u_i\right)\\
&= \frac{1}{h}\left(|S| - \sum_{i=h+1}^{d} u_i - (k-h)t\right)\\
&= \frac{1}{h}\left(|S|-(d-h)-(k-h)t\right)\\
&= \frac{1}{h}(|S|-kt)+t+\left(1-\frac{d}{h}\right)\\
&\geq \frac{1}{k-1}(|S|-kt)+t\\
&= \frac{|S|}{k-1}-\frac{t}{k-1} \geq \frac{|S|}{k-1}-\frac{|G|-k-1}{(k-1)f(G_0,k)}.
\end{aligned}$$

∎

Corollary 3.5. *Let $G = C_n$ with $n \ge 4$ and $S \in \mathcal{F}(G)$ a zerofree sequence. If $|S| \ge \frac{n+3}{2}$, then there exists some $g \in G$ with*

$$v_g(S) \ge max\{2|S| - n + 1\,,\ \frac{|S|}{2} - \frac{n-4}{12}\} \ge \frac{n}{6} + \frac{13}{12}.$$

Proof. One part follows from Lemma 3.3. For the remaining part set $G_0 = \{g \in G | v_g(S) > 0\}$. Assume to the contrary that G_0 contains an element g_0 of order 2. Then n is even and we consider the canonical homomorphism $\varphi : C_n \to C_{n/2}$. Since $|g_0^{-1}S| \ge \frac{n}{2} + \frac{1}{2}$ there exists a proper subsequence T of $g_0^{-1}S$ such that $\varphi(T)$ has sum zero. This implies that g_0T is a proper zero subsequence of S, a contradiction.

Therefore Lemma 3.2 implies that $f(G_0, 3) \ge 6$ and we apply Theorem 3.4 with $k = 3$. Since

$$|S| \ge \frac{n+3}{2} \ge \big(\frac{n-3}{f(G_0,3)} + 1\big) \cdot 3$$

Theorem 3.4 yields that there is some $g \in G$ with

$$v_g(S) \ge \frac{|S|}{2} - \frac{n-4}{2f(G_0,3)} \ge \frac{|S|}{2} - \frac{n-4}{12} \ge \frac{n}{6} + \frac{13}{12}.$$ ∎

We give an example where the above inequality cannot be improved. Let $2 \le k \in \mathbb{N}_+, n = 6k+1$ and

$$S = (1 + n\mathbb{Z})^{k+2}(2 + n\mathbb{Z})^{k+2}(3 + n\mathbb{Z})^{k-2} \in \mathcal{F}(\mathbb{Z}/n\mathbb{Z})\ .$$

Then S is zerofree with $|S| = \frac{n+3}{2}$ and $v_{1+n\mathbb{Z}}(S) = \frac{2n+22}{12}$.

4. On the maximal order of elements

Theorem 4.1. *Let $G = C_n$ be a cyclic group of order $n \ge 4$ and $S \in \mathcal{F}(G)$ a zerofree sequence of length $|S| \ge \frac{n+3}{2}$. Then S contains some element $g \in G$ of order n. Moreover, if $\gcd(6, n) = 1$, then $v_g(S) \ge \frac{n}{6} + \frac{13}{12}$.*

Lemma 4.2. *Let $G = C_n$ with $n \ge 3$, $S \in \mathcal{F}(G)$ a zerofree sequence and m a divisor of n. Let S_1 be the maximal subsequence of S such that $\mathrm{ord}(g) | \frac{n}{m}$ for all $g \in G$ with $v_g(S_1) > 0$. Then $|S_1| < \frac{n-|S|}{m-1}$.*

Proof. Let $\varphi : C_n \to C_m$ denote the canonical epimorphism. Then $\mathrm{Ker}(\varphi) = \{g \in G \mid \mathrm{ord}(g) | \frac{n}{m}\}$ and $S = S_1S_2$ where $S_1 \in \mathcal{F}\big(\mathrm{Ker}(\varphi)\big)$ and $S_2 \in \mathcal{F}\big(G \setminus \mathrm{Ker}(\varphi)\big)$. Set $S_2 = T_1 \ldots T_kT'$ such that all $\varphi(T_i)$ are minimal zero sequences in C_m and

$\varphi(T')$ is zerofree in C_m. Then $|T'| \le m-1$ and for every $1 \le i \le k$ we have $\varphi\big(\iota(T_i)\big) = \iota\big(\varphi(T_i)\big) = 0$ whence $\iota(T_i) \in \mathrm{Ker}(\varphi)$. Therefore

$$S_1 \iota(T_1) \dots \iota(T_k)$$

is a zerofree sequence in $\mathrm{Ker}(\varphi) \cong C_{n/m}$ whence

$$|S_1| + k \le \frac{n}{m} - 1.$$

Since $k \ge \big[\frac{|S|-|S_1|}{m}\big]$ we infer that

$$|S_1| \le \frac{n}{m} - 1 - k < \frac{n}{m} - 1 - \Big(\frac{|S|-|S_1|}{m} - 1\Big) = \frac{n}{m} - \frac{|S|-|S_1|}{m}$$

and thus

$$|S_1| < \frac{n-|S|}{m-1}. \qquad \blacksquare$$

Proof of Theorem 4.1. By Corollary 3.5 there exists some $g \in G$ with $l = v_g(S) \ge \frac{n}{6} + \frac{13}{12}$. Since S is zerofree, this implies that $\mathrm{ord}(g) = \frac{n}{m}$ for some $m \in \{1, \dots, 5\}$. If $m = 1$, then we are done.

Assume to the contrary that $m \in \{4,5\}$. Then by Lemma 4.2 we infer that

$$\frac{2n+13}{12} \le v_g(S) < \frac{n-|S|}{m-1} \le \frac{n-3}{2(m-1)},$$

a contradiction.

Hence it remains to consider the cases that $m \in \{2,3\}$. Assume to the contrary that S contains no element of order n. Since there is an automorphism $\varphi : C_n \to \mathbb{Z}/n\mathbb{Z}$ with $\varphi(g) = m + n\mathbb{Z}$, we may suppose that

$$S = (m+n\mathbb{Z})^l \prod_{i=1}^{r} (ma_i + n\mathbb{Z}) \prod_{i=1}^{s} (b_i + n\mathbb{Z})^m \prod_{i=1}^{m-1} S^{(i)}$$

where, for $1 \le i \le m-1$, $S^{(i)} = \prod_{\nu=1}^{t_i} (mc_\nu^{(i)} + i + n\mathbb{Z})$, $r, s, t_i \in \mathbb{N}$, $a_i, b_i, c_\nu^{(i)} \in \mathbb{Z}, m \nmid b_i$ and no element in $S^{(i)}$ occurs more than $(m-1)$ times. If there is some $i \in \{1, \dots, s\}$ such that $mb_i + n\mathbb{Z} = m + n\mathbb{Z}$, then $b_i + n\mathbb{Z} \in \{j\frac{n}{m} + 1 + n\mathbb{Z} \mid 0 \le j \le m-1\}$ whence $\mathrm{ord}(b_i + n\mathbb{Z}) = n$, a contradiction. Thus it follows that $mb_i + n\mathbb{Z} \ne m + n\mathbb{Z}$ for every $1 \le i \le s$.

Case 1: $m = 3$. Obviously, we can write $S^{(j)}$ as

$$S^{(j)} = S_1^{(j)} \dots S_{u_j}^{(j)} d_1^{(j)} \dots d_{v_j}^{(j)}$$

with $u_j=\left[\frac{t_j-1}{3}\right], 1\le v_j\le 3, |S_i^{(j)}|=3$ and $\iota(S_i^{(j)})\ne 3+n\mathbb{Z}$. Furthermore, there are T, T' such that

$$d_1^{(1)}\dots d_{v_1}^{(1)}d_1^{(2)}\dots d_{v_2}^{(2)}=TT'$$

where $|T|\in\{0,2\}, 3+n\mathbb{Z}\ne\iota(T)\in\langle 3+n\mathbb{Z}\rangle$ and $|T'|\le 4$. Then

$$W=(3+n\mathbb{Z})^l\prod_{i=1}^{r}(3a_i+n\mathbb{Z})\prod_{i=1}^{s}(3b_i+n\mathbb{Z})\prod_{\nu=1}^{u_1}\iota(S_\nu^{(1)})\prod_{\nu=1}^{u_2}\iota(S_\nu^{(2)})\iota(T)$$

is a zerofree sequence in $H=\langle 3+n\mathbb{Z}\rangle<\mathbb{Z}/n\mathbb{Z}$ with $|W|\ge l+\frac{|S|-l-4}{3}$ and $v_{3+n\mathbb{Z}}(W)=l$. Since $\frac{|H|+1}{2}\le l\le |W|$, Lemma 3.3 implies that there is some $h\in H$ with $v_h(W)\ge 2|W|-\frac{n}{3}+1$. Because $2|W|-\frac{n}{3}+1>|W|-l$ we infer that $h=3+n\mathbb{Z}$. However,

$$v_{3+n\mathbb{Z}}(W)=l<2\left(l+\frac{|S|-l-4}{3}\right)-\frac{n}{3}+1\le 2|W|-\frac{n}{3}+1\,,$$

a contradiction.

Case 2: $m=2$. Obviously,

$$S^{(1)}=S_1^{(1)}\dots S_{u_1}^{(1)}d_1^{(1)}\dots d_{v_1}^{(1)}$$

with $0\le v_1\le 1, |S_i^{(1)}|=2$ and $\iota(S_i^{(1)})\ne 2+n\mathbb{Z}$. Then

$$W=(2+n\mathbb{Z})^l\prod_{i=1}^{r}(2a_i+n\mathbb{Z})\prod_{i=1}^{s}(2b_i+n\mathbb{Z})\prod_{\nu=1}^{u_1}\iota(S_\nu^{(1)})$$

is a zerofree sequence in $H=\langle 2+n\mathbb{Z}\rangle<\mathbb{Z}/n\mathbb{Z}$ with $|W|\ge l+\frac{|S|-l-1}{2}$ and $v_{2+n\mathbb{Z}}(W)=l$. Since $\frac{|H|+1}{2}\le |W|$, Lemma 3.3 implies that there is some $h\in H$ with $v_h(W)\ge 2|W|-\frac{n}{2}+1\ge l+\frac{3}{2}$. Thus W has a subsequence $W'=(2+n\mathbb{Z})^l(2a+n\mathbb{Z})^{l+2}$ with $a\in\{2,\dots,\frac{n}{2}-1\}$. However, W' is not zerofree, a contradiction. ∎

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